

Structurally Stable Homoclinic Classes

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Abstract

In this paper we study structurally stable homoclinic classes. In a natural way, the structural stability for an individual homoclinic class is defined through the continuation of periodic points. Since the homoclinic classes is not innately locally maximal, it is hard to answer whether structurally stable homoclinic classes are hyperbolic. In this article, we make some progress on this question. We prove that if a homoclinic class is structurally stable, then it admits a dominated splitting. Moreover we prove that codimension one structurally stable classes are hyperbolic. Also, if the diffeomorphism is far away from homoclinic tangencies, then structurally stable homoclinic classes are hyperbolic.

1 Introduction

An important notion in dynamical systems coming from Physics and Mechanics is the so called structural stability. Precisely, a diffeomorphism f is *structurally stable* if there is a C^1 neighborhood \mathcal{U} of f in $\text{Diff}(M)$ such that, for every $g \in \mathcal{U}$, there is a homeomorphism $h : M \rightarrow M$ such that $h \circ f = g \circ h$. Since such a homeomorphism h preserves orbits, a structurally stable system is one that has robust dynamics, that is, one whose orbital structure remains unchanged under perturbations.

The non-recurrent part of dynamical systems is fairly robust with respect to perturbations. But the recurrent part is fragile and, to survive from perturbations, it needs the condition of (various versions of) hyperbolicity. For instance, a single periodic orbit is structurally stable if and only if it is hyperbolic, meaning no eigenvalue of modulus 1. For the whole system f to be structurally stable, a crucial condition needed is that $\text{CR}(f)$, the set that captures all the recurrence, is a hyperbolic set. Recall a compact invariant set $\Lambda \subset M$ of f is called *hyperbolic* if, for each $x \in \Lambda$, the tangent space $T_x M$ splits into $T_x M = E^s(x) \oplus E^u(x)$ such that

$$Df(E^s(x)) = E^s(f(x)), \quad Df(E^u(x)) = E^u(f(x))$$

and, for some constants $C \geq 1$ and $0 < \lambda < 1$,

$$|Df^n(v)| \leq C\lambda^n|v|, \quad \forall x \in \Lambda, v \in E^s(x), n \geq 0,$$

$$|Df^{-n}(v)| \leq C\lambda^n|v|, \quad \forall x \in \Lambda, v \in E^u(x), n \geq 0.$$

Briefly, a hyperbolic set is one at which tangent vectors split into two directions, contracting and expanding upon iterates, respectively, with uniform exponential rates. This definition extends the hyperbolicity condition from a single periodic orbit to a general compact invariant set. It is closely related to structural stability. Indeed, the following remarkable result, known as the stability conjecture of Palis and Smale [PS1], is fundamental to dynamical systems:

Theorem (Mañé [Man2]). *If a diffeomorphism f is structurally stable then $\text{CR}(f)$ is hyperbolic.*

In this paper we consider a more general version of structural stability. It is for an individual “basic piece” of the dynamics, which is not necessarily manifold M or the whole nonwandering set $\Omega(f)$. A homoclinic class and a chain recurrent class are two typical types of “basic pieces”.

Let M be a compact C^∞ Riemannian manifold without boundary, and $f : M \rightarrow M$ be a diffeomorphism. Denote $\text{Diff}(M)$ the space of diffeomorphisms of M with the C^1 -topology.

Let p and q be hyperbolic periodic points. We say that p and q are *homoclinically related* and write $p \sim q$, if $W^s(\text{Orb}(q))$ (resp. $W^u(\text{Orb}(q))$) and $W^u(\text{Orb}(p))$ (resp. $W^s(\text{Orb}(p))$) have non-empty transverse intersections. The set $H(p, f) = \overline{\{q : q \sim p\}}$ is called the *homoclinic class* of p . Equivalently, $H(p, f)$ is the closure of set of the transversely intersection points of $W^s(\text{Orb}(p))$ and $W^u(\text{Orb}(p))$.

A hyperbolic periodic point has its natural “continuations”. Precisely, let $p \in M$ be a hyperbolic periodic point of f of period k . Then there exist a compact neighborhood U of $\text{Orb}(p)$ in M and a C^1 -neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, the maximal invariant set

$$\bigcap_{n=-\infty}^{\infty} g^n(U)$$

of g in U consists of a single periodic orbit O_g of g of the same period as p , which is hyperbolic with $\text{Ind}(O_g) = \text{Ind}(p)$. Here $\text{Ind}(p)$ denotes the *index* of p , which is the dimension of the stable manifold of p . The neighborhood U can be chosen to be the union of k arbitrarily small disjoint balls, each containing exactly one point of $\text{Orb}(p)$ and one point of O_g . This identifies the *continuation* p_g of p under g . Thus the notion of continuation p_g of p is defined for g sufficiently close to f . As usual, the (unique) homoclinic class of g that contains p_g is denoted $H(p_g, g)$. Here is a natural general version of structural stability for a single homoclinic class:

Definition 1.1. *Let p be a hyperbolic periodic point of f . We say that $H(p, f)$ is C^1 -structurally stable if there is a neighborhood \mathcal{U} of f in $\text{Diff}(M)$ such that, for every $g \in \mathcal{U}$, there is a homeomorphism $h : H(p, f) \rightarrow H(p_g, g)$ such that $h \circ f|_{H(p, f)} = g \circ h|_{H(p, f)}$, where p_g is the continuation of p .*

Note that, while h in this definition preserves periodic points of $H_f(p)$, it is not clear if it preserves individual continuations. For instance, it is not clear if $h(p) = p_g$. Also, since h is a homeomorphism only, it does not preserve the hyperbolicity of periodic points.

In the article, we get some characterization of the structurally stable homoclinic classes as follows:

Theorem 1.1. *Let $H(p, f)$ be a structurally stable homoclinic class of f , then there are constants $C > 1$ and $0 < \lambda < 1$ and an integer $m > 0$ such that*

1. $H(p, f)$ admit a dominated splitting $T_{H(p, f)}M = E \oplus F$ with $\dim E = \text{ind}(p)$.

2. For any $q \sim p$,

$$\prod_{i=0}^{k-1} \|Df^m|_{E^s(f^{im}(q))}\| < C\lambda^k,$$

$$\prod_{i=0}^{k-1} \|Df^{-m}|_{E^u(f^{-im}(q))}\| < C\lambda^k,$$

where $k = [\pi(q)/m]$ ($\pi(q)$ represents the minimal period of q and $[\cdot]$ represents the integer part).

Here we recall that an invariant set Λ of f admits a dominated splitting if the tangent bundle $T_\Lambda M$ has a continuous Df -invariant splitting $E \oplus F$ and there exists constants $C > 0$, $0 < \lambda < 1$ such that

$$\|Df^n|_{E(x)}\|/m(Df^n|_{F(x)}) \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. Where

$$m(A) = \inf\{\|Av\| : \|v\| = 1\}$$

denotes the *mininorm* of a linear map A . The dominated splitting is a generalization (or a candidate) of hyperbolic splitting. With some assumptions, we can prove that the dominated splitting given in Theorem 1.1 is actually hyperbolic.

Theorem 1.2. *Let f be a diffeomorphism of M and p be a hyperbolic periodic point of f of index 1 or $\dim M - 1$. If the homoclinic class $H(p, f)$ of p is structurally stable, then $H(p, f)$ is hyperbolic.*

Recall that a diffeomorphism f is called *far away from homoclinic tangency* if there is a neighborhood \mathcal{U} of f such that for any $g \in \mathcal{U}$ and any hyperbolic periodic points q of g , there is no non-transversely intersection of $W^s(q)$ and $W^u(q)$.

Theorem 1.3. *Let f be a diffeomorphism of M which is far away from homoclinic tangency and p be a hyperbolic periodic point of f . If the homoclinic class $H(p, f)$ of p is structurally stable, then $H(p, f)$ is hyperbolic.*

2 Periodic points in $H(p, f)$

In this section we give the proof of Theorem 1.1. To prove Theorem 1.1 we need consider a kind of typical diffeomorphisms with some generic properties.

Proposition 2.1. *There is a residual subset $\mathcal{R}_0 \subset \text{Diff}(M)$ such that every $f \in \mathcal{R}_0$ satisfies the following conditions:*

1. *f is Kupka-Smale, meaning periodic points of f are each hyperbolic and their stable and unstable manifolds meet transversally (see [PM]).*
2. *for any pair of hyperbolic periodic points p and q of f , either $H(p, f) = H(q, f)$ or $H(p, f) \cap H(q, f) = \emptyset$. (see [BC])*
3. *if two hyperbolic periodic points p and q of f are in the same topologically transitive set and $\text{Ind}(p) \leq \text{Ind}(q)$, then $W^s(\text{Orb}(q), f) \pitchfork W^u(\text{Orb}(p), f) \neq \emptyset$ (see [GW]).*
4. *for every pair of periodic points p_f and q_f of f , there exists a neighborhood \mathcal{U} of f in \mathcal{R}_0 such that either $H(p_g, g) = H(q_g, g)$ for all $g \in \mathcal{U}$ or $H(p_g, g) \cap H(q_g, g) = \emptyset$ for all $g \in \mathcal{U}$ (see [ABCDW]).*

In the proof we also need the following version of Frank's Lemma preserving (un)stable manifolds.

Proposition 2.2. ([Gou]) *Let f be a diffeomorphism of M . For any C^1 neighborhood \mathcal{U} of f , there is $\epsilon > 0$ such that, for any pair of hyperbolic periodic points $p, q \in M$ of f that are homoclinically related, any neighborhood U of $\text{Orb}(q)$ in M not touching $\text{Orb}(p)$, and any continuous path of linear isomorphisms $A_{k,t} : T_{f^k q} M \rightarrow T_{f^{k+1} q} M$ that satisfies the following three assumptions:*

- (1) $A_{k,0} = D_{f^k q} f$ for all $0 \leq k < \pi(q)$,
- (2) $\|A_{k,t} - D_{f^k(q)} f\| < \epsilon$ for all $0 \leq k < \pi(q)$ and any $t \in [0, 1]$,
- (3) $A_{\pi(q)-1,t} \circ A_{\pi(q)-2,t} \circ \cdots \circ A_{0,t}$ has no eigenvalue on the unit circle for all $t \in [0, 1]$,

there exist a perturbation $g \in \mathcal{U}$ with the following three properties:

- (A) $g = f$ on $(M \setminus U) \cup \text{Orb}(q)$,
- (B) $D_{f^k q} g = A_{k,1}$ for all $0 \leq k < \pi(q)$,
- (C) p and q are homoclinically related with respect to g .

Let q be a hyperbolic periodic point of a diffeomorphism f . We say an eigenvalue μ of $D_q f^{\pi(q)}$ is the *weakest stable eigenvalue* of q if $\log |\mu'| \leq \log |\mu| < 0$ for all eigenvalues μ' of $D_q f^{\pi(q)}$ with $|\mu'| < 1$. Similarly we define the weakest unstable eigenvalue. We say ${}^{\pi(q)}\sqrt{|\mu|}$ is the *normalized weakest stable eigenvalue* of q . We also say an eigenvalue is simple if it has multiplicity 1. Applying the above proposition, we can get the following lemma.

Lemma 2.1. *Let f be a diffeomorphism of M and $q \sim p$ be hyperbolic periodic points of f . Let μ be a weakest stable eigenvalue of q w.r.t g . Then there exists a diffeomorphism g arbitrarily close to f with a hyperbolic periodic point $q' \sim p_g$ such that the weakest stable eigenvalue μ' of q' (w.r.t g) is real and ${}^{\pi(q')}\sqrt{|\mu'|}$ is arbitrarily close to ${}^{\pi(q)}\sqrt{|\mu|}$.*

Proof. Since $q \sim p$, one can find points $x \in W^s(p) \pitchfork W^u(q)$ and $y \in W^u(p) \pitchfork W^s(q)$. It follows that $\text{Orb}(p) \cup \text{Orb}(x) \cup \text{Orb}(q) \cup \text{Orb}(y)$ is a hyperbolic set. Applying the shadowing lemma of hyperbolic set, one can find a hyperbolic periodic point q' of f such that the period of q' is arbitrarily large and

the normalized weakest stable eigenvalue of q' is arbitrarily close to $\pi^{(q)}\sqrt{|\mu|}$. Without loss of generality, we assume that the weakest stable eigenvalue of q' is complex. For any $\varepsilon > 0$, by choosing a suitable q' with period large enough, using the Lemma 6.6 of [BC], one can get continuous paths of linear isomorphisms $A_{k,t} : T_{f^k q'} M \rightarrow T_{f^{k+1} q'} M$ ($k = 0, 1, \dots, \pi(q') - 1$) with the following properties:

- (1) $A_{k,0} = D_{f^k q'} f$ for all $0 \leq k < \pi(q')$,
- (2) $\|A_{k,t} - D_{f^k(q')} f\| < \varepsilon$ for all $0 \leq k < \pi(q')$ and any $t \in [0, 1]$,
- (3) $A_{\pi(q')-1,t} \circ A_{\pi(q')-2,t} \circ \dots \circ A_{0,t}$ has no eigenvalue on the unit circle for all $t \in [0, 1]$,
- (4) The weakest stable eigenvalue of $A_{\pi(q')-1,1} \circ A_{\pi(q')-2,1} \circ \dots \circ A_{0,1}$ are real and simple and the normalization of the eigenvalue equal to the normalized weakest stable eigenvalue of q' .

Then one can apply the Proposition 2.2 to get the perturbation g in the lemma.

Let p be a hyperbolic periodic point of f . If $q \in H(p, f)$ has a simple real stable weakest eigenvalue λ , then we know that there is a strong stable manifold $W_{loc}^{ss}(q)$ of codimension one in $W_{loc}^s(q)$. If we choose loc small enough, then $W_{loc}^s(q) \setminus W_{loc}^{ss}(q)$ has two components B_1 and B_2 . We say that q is a *non-ss boundary point* if $W^u(p)$ intersect both B_1 and B_2 . The following lemma can be found in [SV]. Here we also give a proof for completeness.

Lemma 2.2. *Let $f \in \text{Diff}(M)$ with a neighborhood $\mathcal{U} \subset \text{Diff}(M)$ and p be a hyperbolic periodic point of f . There exist a constant $\delta > 0$ such that for any $q \in H(p, f)$, if q has a simple real weakest stable eigenvalue λ with $(1 - \delta)^{\pi(q)} < |\lambda| < 1$ and q is a non-ss boundary point, then for any neighborhood U of $\text{Orb}(q)$, there exist a diffeomorphism $g \in \mathcal{U}$ such that*

1. $g = f$ in $M \setminus U$,
2. *there exist two hyperbolic periodic points q_1, q_2 of g which are homoclinic related to p such that their orbits contained in U and $\pi(q_1) = \pi(q_2)$. Furthermore, the periods of q_1, q_2 are either $\pi(q)$ or $2\pi(q)$.*

Proof. After an arbitrarily small perturbation, we can assume that f is “locally linear” near the orbit of q in the sense that there is $r > 0$ such that

$$f|_{B_r(f^i q)} = \exp_{f^{i+1} q} \circ D_{f^i q} g \circ \exp_{f^i q}^{-1}$$

for any $0 \leq i < \pi(q)$. Let $E^c(f^i q) \subset T_{f^i q} M$ be the eigenspace of $D_q g^{\pi(q)}$ associated to the simple real weakest stable eigenvalue λ and $E^{ss}(f^i q)$ be the strong stable eigenspace with $E^c(f^i q) \oplus E^{ss}(f^i q) = E^s(f^i q)$. For $\eta > 0$, denote the ball in $E^c(f^i q)$ of radius η about the origin to be $E^c(f^i q, \eta)$. Similarly we define $E^s(f^i q, \eta)$ and $E^{ss}(f^i q, \eta)$. Choose r_0 small enough such that $E^s(f^i q, r_0) \subset W^s(f^i q)$ and $E^{ss}(f^i q, r_0) \subset W^{ss}(f^i q)$. Let B_1, B_2 be the two components of $E^s(q, r_0) \setminus E^{ss}(q, r_0)$. By the assumption we know that there exist $y_1, y_2 \in W^u(p)$ such that $y_1 \in B_1$ and $y_2 \in B_2$ which are the transverse intersection points of $W^u(p)$ and $W^s(q)$.

We construct a perturbation \tilde{g} of f . Let $\alpha(x) : [0, +\infty) \rightarrow [0, +\infty)$ be a bump function which satisfies (1) $\alpha|_{[0, 1/3]} = 1$, (2) $\alpha|_{[2/3, +\infty)} = 0$, (3) $0 < \alpha|_{(1/3, 2/3)} < 1$ and (4) $0 \leq \alpha'(x) < 4$ for all $x \in [0, +\infty)$. For a small $\eta > 0$ and any $0 \leq i < \pi(q)$, let $\xi = \sqrt[\pi(q)]{|\lambda|}$ be the normalized eigenvalue associated to λ , define a real function $\beta : T_{f^i q} M \rightarrow \mathbb{R}$ by

$$\beta(v) = \xi^{-1} \alpha(|v|/\eta) + (1 - \alpha(|v|/\eta)).$$

Thus $\beta(v) = \xi^{-1}$ for $|v| \leq \eta/3$, $1 < \beta(v) < \xi^{-1}$ for $\eta/3 < |v| < 2\eta/3$, and $\beta(v) = 1$ for $|v| \geq 2\eta/3$. We always assume η much less than r_0 . Define a perturbation \tilde{g} of f in $\bigcup_{0 \leq i < \pi(q)} B(f^i q, \eta)$ to be

$$\tilde{g}(x) = \exp_{f^{i+1} q}(\beta(v) \cdot D_{f^i q} f(v)), \quad v = \exp_{f^i q}^{-1}(x)$$

for $x \in B(f^i q, \eta)$ for all $0 \leq i < \pi(q)$, and define $\tilde{g}(x) = f(x)$ for $x \notin \bigcup_{0 \leq i < \pi(q)} B(f^i q, \eta)$. It is easy to check that \tilde{g} is C^1 close to f if ξ is sufficiently close to 1.

One can easily check that $\tilde{g}^{\pi(q)}|_{\exp_q(E^c(q, \eta/3))} = Id$ or $-Id$. Let q_1, q_2 be the end points of $\exp_q(E^c(q, \eta/3))$. We know that q_1, q_2 are periodic points of \tilde{g} with period equal to $\pi(q)$ or $2\pi(q)$. If we choose η small enough, the

negative orbits of $y_{1,2}$ will be unchanged under the perturbation \tilde{g} and the positive orbits of $y_{1,2}$ will go to the orbits of $q_{1,2}$ respectively. We also know that if η is small then the local unstable manifolds of $q_{1,2}$ (with respect to \tilde{g}) will be C^1 close to the local unstable manifold of q (with respect to f). Since the original $W^u(q)$ (with respect to f) has a transverse intersection with $W^s(p)$, one has that $W^u(q_{1,2})$ (with respect to \tilde{g}) will have a transverse intersection with $W^s(p)$. After an arbitrarily small perturbation near $q_{1,2}$, we can make $q_{1,2}$ hyperbolic and $y_{1,2}$ still in the intersection of $W^u(p)$ and $W^s(q_{1,2})$ respectively. This shows the existence of g and ends the proof of lemma.

Now we prove that, in a structurally stable homoclinic class, eigenvalues of periodic orbits are uniformly and robustly away from the unit circle.

Proposition 2.3. *Let f be a diffeomorphism and let $p \in M$ be a hyperbolic periodic point of f . If $H(p, f)$ is structurally stable, then there are a constant $0 < \alpha < 1$ and a neighborhood \mathcal{U} of f such that, for any $g \in \mathcal{U}$ and any periodic point q of g that is homoclinically related to p_g , the derivative $D_q g^{\pi(q)}$ has no eigenvalue with modulus in $(\alpha^{\pi(q)}, \alpha^{-\pi(q)})$.*

Proof. We prove by contradiction. Suppose there is a diffeomorphism g arbitrarily C^1 close to f and a periodic point $q_0 \in H(p_g, g)$ homoclinically related to p_g such that $D_{q_0} g^{\pi(q_0)}$ has an eigenvalue μ with $|\mu|^{\frac{1}{\pi(q_0)}}$ arbitrarily close to 1.

Without loss of generality we assume that $g \in \mathcal{R}_0$ and the eigenvalue μ of q_0 is the weakest stable eigenvalue. By using Lemma 2.1, we also can assume that μ is real and has multiplicity 1.

Let $W_{loc}^{ss}(q_0)$ be the codimension one strong stable manifold of q_0 with respect to g . Choose loc small enough such that $W_{loc}^s(q_0) \setminus W_{loc}^{ss}(q_0)$ has two components B_1 and B_2 . We can assume that q_0 is a non-ss boundary point. Otherwise, if q_0 is not a non-ss boundary point, without loss of generality, we can assume that $W^u(p)$ transversely intersects B_1 at the point z . By the λ -Lemma we can find a point $s \in W^s(q_0) \cap W^u(q_0)$ which is arbitrarily close

to z . It is well known that $\text{Orb}(q_0) \cup \text{Orb}(s)$ is a hyperbolic invariant set of g . Replacing g by an arbitrarily small perturbation we can assume that $\text{Orb}(q_0) \cup \text{Orb}(s)$ also admits a partially hyperbolic splitting $E^{ss} \oplus E^{cs} \oplus E^u$ where $E^{cs}(q_0)$ is the eigenspace of $D_{q_0}g^{\pi(q_0)}$ associated to the eigenvalue μ (See Lemma 4.13 of [BDP] for the construction of the perturbation). By taking $k > 0$ large one can assume that $W_{loc}^{ss}(g^{k\pi(q_0)}s)$ is closer to $W_{loc}^{ss}(q_0)$ than $W_{loc}^{ss}(s)$. By the shadowing property of $\text{Orb}(q_0) \cup \text{Orb}(s)$, one can find a hyperbolic periodic $q'_0 \sim q_0$ arbitrarily close to $f^{k\pi(q_0)}s$ such that its orbit stays mostly nearby the orbit of q_0 . Since the orbit of q'_0 is chosen close to the partially hyperbolic set $\text{Orb}(q_0) \cup \text{Orb}(s)$, the strong stable manifold $W_{loc}^{ss}(q'_0)$ is well defined and so $W_{loc}^s(q'_0) \setminus W_{loc}^{ss}(q'_0)$ has two component $B_1(q_0)$ and $B_2(q_0)$. We can choose q'_0 close to $f^{k\pi(q_0)}s$ enough such that $W^u(q_0)$ crosses $B_1(q_0)$ and $W^u(s)$ crosses $B_2(q_0)$. By the λ -Lemma we know that q'_0 is a non-ss boundary point. If the orbit of q'_0 spends enough time nearby the orbit of q_0 we can get that q'_0 has a real simple normalized weakest eigenvalue close to ${}^{\pi(q_0)}\sqrt{|\mu|}$. In this case we replace q_0 by q'_0 .

Now we go back to the proof of proposition. We can separate the hyperbolic periodic points in $H(p_g, g)$ into several sets. Let I be the set of integers i such that there exist hyperbolic periodic points of index i in $H(p_g, g)$. Assume $I = \{i_1, i_2, \dots, i_s\}$. For any $i_k \in I$, denote

$$P^k = \{q \in H(p_g, g) | q \text{ is a periodic point of index } i_k\}.$$

Choose $p_k \in P^k$ such that p_k has the smallest period in P^k . By item 4 of Proposition 2.1, we can find a neighborhood $\mathcal{V} \subset \mathcal{R}_0$ of g such that for any $g' \in \mathcal{V}$, $H(p_{g'}, g') = H(p_{i_{g'}}, g')$ for all $i = 1, 2, \dots, s$. Note here that $p_{g'}$ is the continuation of p and $p_{i_{g'}}$ is the continuation of p_i associated to g' .

Without loss of generality, we assume that $p \in P^1$ and hence q_0 is also contained in P^1 . Let

$$N = \max\{\pi(p_1), \pi(p_2), \dots, \pi(p_s), 2\pi(q_0)\}$$

and $P_N(g) = \{q \in H(p_g, g) | q \text{ is a periodic point with } \pi(q) \leq N\}$ and $P_N^k = P_N(g) \cap P^k$. It is easy to see that $P_N(g)$ and P_N^k are finite set. Let $P_N^1 =$

$\{p_1^1, \dots, p_{l_1}^1\} \cup \text{Orb}(q_0)$ and $P_N^k = \{p_1^k, \dots, p_{l_k}^k\}$ for $k = 2, \dots, s$. By item 3 of Proposition 2.1 one can get that $p_i^k \sim p_j^k$ for all $1 \leq i < j \leq l_k$ and $k = 1, 2, \dots, s$. Pick $x_{kij} \in W^s(\text{Orb}(p_i^k)) \cap W^u(\text{Orb}(p_j^k))$ and $y_{kij} \in W^u(\text{Orb}(p_i^k)) \cap W^s(\text{Orb}(p_j^k))$ for all $1 \leq k \leq s$ and $1 \leq i < j \leq l_k$. Let Γ be the union of all orbits of p_i^k and all orbits of x_{kij} and all orbits of y_{kij} . We know that Γ is a close set and $\Gamma \cap \text{Orb}(q_0) = \emptyset$. Now we take a neighborhood U of $\text{Orb}(q_0)$ such that $\Gamma \cap U = \emptyset$ and perform a perturbation g' by Lemma 2.2. With another arbitrarily small perturbation we can assume that $g' \in \mathcal{V}$. Let $P_N(g') = \{q \in H(p_{g'}, g') | q \text{ is a periodic point of } g' \text{ with } \pi(q) \leq N\}$. Since $g' \in \mathcal{V}$, we know that $p_{i_{g'}} \in H(p_{g'}, g')$ and all $p_{i_{g'}}^k \in H(p_{g'}, g')$. The periodic point q_0 of g disappears in $H(p_{g'}, g')$ but two more periodic points q_1, q_2 appear in $H(p_{g'}, g')$. One can easily get that $\sharp P_N(g') \geq \sharp P_N(g) + 1$. This contradict that $H(p_g, g)$ is conjugate to $H(p_{g'}, g')$. This ends the proof of Proposition 2.3.

Let $\pi : E \rightarrow \Lambda$ be a finite dimensional vector bundle and $f : \Lambda \rightarrow \Lambda$ be a homeomorphism. A continuous map $A : E \rightarrow E$ is called a *linear co-cycle* (or *bundle isomorphism*) if $\pi A = f\pi$, and if A restricted to every fiber is a linear isomorphism. The topology of Λ is not relevant to our aim here, and we assume that Λ has the discrete topology. We say A is *bounded* if there is $N > 0$ such that $\max\{\|A(x)\|, \|A^{-1}(x)\|\} \leq N$ for every $x \in \Lambda$, where $A(x)$ denotes $A|_{E(x)}$. For two linear co-cycles A and B over the same base map $f : \Lambda \rightarrow \Lambda$, define

$$d(A, B) = \sup_{x \in \Lambda} \{\|A(x) - B(x)\|, \|A^{-1}(x) - B^{-1}(x)\|\}.$$

A periodic point $p \in \Lambda$ of f is called *hyperbolic* with respect to A if $A^{\pi(p)}$ has no eigenvalues of absolute value 1, where $\pi(p)$ is the period of p . As usual, we denote the contracting and expanding subspaces of p to be $E^s(p)$ and $E^u(p)$. Then $E(p) = E^s(p) \oplus E^u(p)$. If every point in Λ is periodic of f , then A is called a *periodic linear co-cycle*. A bounded periodic linear co-cycle A is called a *star system* if there is $\epsilon > 0$ such that any B with $d(B, A) < \epsilon$ has no non-hyperbolic periodic orbits. (This notion corresponds

to that of diffeomorphisms on the manifold M but, since perturbations on manifolds are less restrictive, the star condition on manifolds is stronger. In fact a star condition on a manifold implies Axiom A and no-cycle.) The next fundamental result of Liao and Mañé says that, if A is a star system, then the individual hyperbolic splittings $E^s(p) \oplus E^u(p)$ of $p \in \Lambda$, put together, form a dominated splitting. It also gives some estimates for rates on periodic orbits.

Theorem 2.4. (*[Liao1], [Man]*) *Let $A : E \rightarrow E$ be a bounded periodic linear co-cycle over $f : \Lambda \rightarrow \Lambda$. If A is a star system, then there is $\epsilon > 0$ and three constants $m > 0$, $C > 0$ and $0 < \lambda < 1$ such that, for any linear co-cycle B over f with $d(B, A) < \epsilon$, and any periodic point q of B , the following conditions are satisfied:*

- (1) $\|B^m|_{E^s(q)}\| \cdot \|B^{-m}|_{E^u(f^m q)}\| < \lambda$.
- (2) Let $k = [\pi(q)/m]$, then

$$\prod_{i=0}^{k-1} \|B^m|_{E^s(f^{im}(q))}\| < C\lambda^k,$$

$$\prod_{i=0}^{k-1} \|B^{-m}|_{E^u(f^{-im}(q))}\| < C\lambda^k.$$

The two inequalities in Item 2 are usually referred to as “uniformly contracting (expanding) at the periods” for periodic orbits. We remark that Liao and Mañé did not use the term “linear co-cycles”. Liao worked (for flows) on tangent bundles of manifolds, and Mañé worked on periodic sequences of linear isomorphisms.

Let Λ be the union of periodic orbits of $H(p, f)$ which are homoclinically related to p . The tangent map $Df : T_\Lambda M \rightarrow T_\Lambda M$ acts as a periodic linear co-cycle over f . We verify that it is a star system in the sense of linear co-cycles hence Theorem 1.1 will follow.

Proof of Theorem 1.1. Suppose for the contrary there is a linear co-cycle $A : T_\Lambda M \rightarrow T_\Lambda M$ arbitrarily close to Df that has a periodic orbit $\text{Orb}(q)$ of

f which is non-hyperbolic with respect to A . We join A with Df by a path A_t with $A_0 = Df$ and $A_1 = A$. Since A can be arbitrarily close to Df , we may assume $A_t|_{\text{Orb}(q)}$ satisfies assumption (2) of Proposition 2.2, for every $t \in [0, 1]$. Let $s \in (0, 1]$ be the first parameter that makes q non-hyperbolic, namely, q is non-hyperbolic with respect to A_s , but is hyperbolic with respect to A_t , for every $t \in [0, s)$. Take s' slightly less than s so that one of the eigenvalues μ of $A_{\pi(q)-1, s'} \circ A_{\pi(q)-2, s'} \circ \cdots \circ A_{0, s'}$ (in absolute value) is within $(\alpha^{\pi(q)}, \alpha^{-\pi(q)})$. Then the path A_t , $t \in [0, s']$, satisfies the three assumptions of Proposition 2.2, hence there is $g \in \mathcal{U}$ that preserves $\text{Orb}(q)$ and $\text{Orb}(p)$ such that $Dg|_{\text{Orb}(q)} = A_{s'}|_{\text{Orb}(q)}$ and such that p and q are homoclinically related with respect to g . Such a weak eigenvalue μ contradicts Proposition 2.3. This verifies that $Df : T_\Lambda M \rightarrow T_\Lambda M$ is a star periodic linear co-cycle over f . Thus Theorem 1.1 follows from Theorem 2.4.

3 Structurally stable homoclinic class of generic diffeomorphisms

In this section, we will prove a weak version of Theorem 1.2 and Theorem 1.3 under the generic assumptions.

Proposition 3.1. *There exists a residual set $\mathcal{R}_1 \subset \text{Diff}(M)$ such that for any $f \in \mathcal{R}_1$ and any hyperbolic periodic point p of f with $\text{Ind}(p) = 1$, the structural stability of $H(p, f)$ implies the hyperbolicity.*

Proof. The proof of the above proposition is just as the same as the proof of the Proposition 5.1 of [WW]. For completeness we give a sketch of the proof here. Assuming $f \in \text{Diff}(M)$ and $\text{Ind}(p) = 1$, from Theorem 1.1 we know that if the homoclinic class $H(p, f)$ is structurally stable then the class admit a dominated splitting $T_{H(p, f)}M = E \oplus F$ with $\dim E = 1$. It is proved in [WW] and [Cro2] that there is a residual set $\mathcal{R}_1 \subset \text{Diff}(M)$ such that if $f \in \mathcal{R}_1$ and the one dimensional bundle E is not contracting, then for a minimally non-contracting (w.r.t the bundle E) set $\Lambda \subset H(p, f)$, the

dominated splitting $T_\Lambda M = E \oplus F$ will be partially hyperbolic and

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df|_{E(f^i x)}\| = 0$$

for any $x \in \Lambda$. Then by applying Theorem 4.1 of [WW] we know that for this partially hyperbolic set Λ and any neighborhood U of Λ , one can find a periodic orbit O contained in $U \cap H(p, f)$ of index 1. Thus we can take a sequence of periodic orbits $Q_n \subset H(p, f)$ such that $Q_n \rightarrow \Lambda$ in the Hausdorff metric. By Proposition 2.1, each Q_n is homoclinically related to $\text{Orb}(p)$. By Theorem 1.1, there exist $\lambda \in (0, 1)$ and a positive integer m such that

$$\prod_{i=0}^{k-1} \|Df^m|_{E^c(f^{im}(q))}\| < \lambda^k$$

for $q \in Q_n$, where $k = [\pi(q)/m]$. Note that since K is non-trivial and hence $\pi(Q_n) \rightarrow \infty$, by slightly enlarging λ if necessary we may assume $C = 1$ in the inequality. Take $\lambda' \in (\lambda, 1)$. By Pliss's Lemma, there are $q_n \in Q_n$ such that

$$\prod_{i=0}^{j-1} \|Df^m|_{E^c(f^{im}(q_n))}\| < (\lambda')^j$$

for all $j \geq 1$ (One can also find the proof in the proof of Lemma 4.2). Taking a subsequence if necessary we assume $q_n \rightarrow x \in \Lambda$. Then

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m|_{E^c(f^{im}(x))}\| < \log \lambda'.$$

This contradicts the above limit equality. Thus E is contracting. Then one can apply the main theorem of [BGY] to get the hyperbolicity of $H(p, f)$.

In [CSY], the authors proved that if $H(p, f)$ is a homoclinic class of a diffeomorphism far away from homoclinic tangency, then $H(p, f)$ admits a partially hyperbolic splitting. Furthermore, they proved the following conclusion as stated in Corollary 1.4 of [CSY].

Proposition 3.2. *For generic $f \in \text{Diff}(M)$ which is far away from homoclinic tangencies, the homoclinic classes $H(p, f)$ of f satisfy:*

- *either $H(p, f)$ is hyperbolic,*
- *or $H(p, f)$ contains weak periodic orbits related to p .*

From the above proposition and the Theorem 1.1, one can easily get the following proposition.

Proposition 3.3. *There exists a residual subset $\mathcal{R}_2 \subset \text{Diff}(M)$ such that for any $f \in \mathcal{R}_2$ which is far away from homoclinic tangencies and any hyperbolic periodic point p of f , if the homoclinic class $H(p, f)$ of p is structurally stable, then $H(p, f)$ is hyperbolic.*

4 The proof of Theorem 1.2 and Theorem 1.3

From Proposition 3.1 and Proposition 3.3, assuming $H(p, f)$ is structurally stable, we know that if the index of p is 1 or $\dim M - 1$ or the diffeomorphism f is far way from homoclinic tangencies, then for any neighborhood $\mathcal{U} \subset \text{Diff}(M)$ of f , there exists $g \in \mathcal{U}$ such that $H(p_g, g)$ is hyperbolic.

Lemma 4.1. *Let $f \in \text{Diff}(M)$ and $H(p, f)$ be a structurally stable homoclinic class of f . If for any neighborhood $\mathcal{U} \subset \text{Diff}(M)$ of f , there exist $g \in \mathcal{U}$ such that $H(p_g, g)$ is hyperbolic, then $H(p, f)$ satisfies the shadowing property. Actually, any periodic pseudo orbit in $H(p, f)$ is shadowed by periodic orbit contained in $H(p, f)$.*

Proof. Let $h : H(p, f) \rightarrow H(p_g, g)$ be the conjugate homeomorphism. For any $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that for any $x_1, x_2 \in H(p_g, g)$, if $d(x_1, x_2) < \varepsilon'$, then $d(h^{-1}(x_1), h^{-1}(x_2)) < \varepsilon$. Since $H(p_g, g)$ is hyperbolic, it is locally maximal and admits the shadowing property. Hence for $\varepsilon' > 0$, there exist $\delta' > 0$ such that any δ' -pseudo orbit of $g|_{H(p_g, g)}$ can be ε' shadowed by a real orbit of g in $H(p_g, g)$. There exists $\delta > 0$ such that for any $y_1, y_2 \in H(p, f)$, if $d(y_1, y_2) < \delta$, then $d(h(y_1), h(y_2)) < \delta'$.

Now we let $\{y_i\}_{i=a}^b$ ($-\infty \leq a < b \leq +\infty$) be a δ -pseudo orbit of $f|_{H(p,f)}$. By the choice of δ we know that $\{h(y_i)\}$ should be a δ' -pseudo orbit of $g|_{H(p_g,g)}$. Then we know that there exists $x \in H(p_g, g)$ such that $d(g^i(x), h(y_i)) < \varepsilon'$ for all $a \leq i \leq b$. By the choice of ε' , we know that $d(f^i(h^{-1}x); y_i) < \varepsilon$ for all $a \leq i \leq b$. From the hyperbolicity of $H(p_g, g)$ we also know that if the δ -pseudo orbit $\{y_i\}$ of $f|_{H(p,f)}$ is periodic then the shadowing points $h^{-1}(x)$ is also periodic.

From the above lemma one can see that Theorem 1.2 and Theorem 1.3 is a direct corollary of the following proposition.

Proposition 4.1. *Let p be a hyperbolic periodic point, and $H(p, f)$ be the homoclinic class of f containing p . Assume there exist constants $N \in \mathbb{N}, 0 < \lambda < 1$ such that $H(p, f)$ satisfies the following properties (P1) to (P3).*

(P1). *There exists a continuous Df -invariant splitting $T_{H(p,f)}M = E \oplus F$ with $\dim E = \text{ind}(p)$ such that for every $x \in \Lambda$,*

$$\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2.$$

(P2). *For any hyperbolic periodic point q , if q is homoclinic related to p with period $\pi(q) > N$, then*

$$\prod_{i=0}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| < \lambda^{\pi(q)}$$

$$\prod_{i=0}^{\pi(q)-1} \|Df^{-1}|_{E^u(f^{-i}(q))}\| < \lambda^{\pi(q)}.$$

(P3). *$f|_{\Lambda}$ has the shadowing property and every periodic pseudo orbit can be shadowed by a periodic orbit.*

Then $H(p, f)$ is hyperbolic for f .

To prove the proposition we prepare some lemmas. The following lemma is a folklore.

Lemma 4.2. *Let $0 < \lambda < 1$ be given and let q be a hyperbolic periodic point of f with $\|Df|_{E^s(x)}\|/m(Df|_{E^u(x)}) < \lambda^2$ for any $x \in \text{Orb}(q)$. If*

$$\prod_{i=0}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| < \lambda^{\pi(q)},$$

$$\prod_{i=0}^{\pi(q)-1} \|Df^{-1}|_{E^u(f^{-i}(q))}\| < \lambda^{\pi(q)},$$

then there exists $q' \in \text{Orb}(q)$ such that

$$\prod_{i=0}^{k-1} \|Df|_{E^s(f^i(q'))}\| \leq \lambda^k,$$

$$\prod_{i=0}^{k-1} \|Df^{-1}|_{E^u(f^{-i}(q'))}\| \leq \lambda^k,$$

for any $k \geq 1$.

Proof. Firstly we can prove that there exist $n \in \mathbb{N}$ such that

$$\prod_{i=0}^{k-1} \|Df|_{E^s(f^i(f^n q))}\| \leq \lambda^k$$

for any $k \geq 1$. Otherwise, there exist $k_j \geq 1$ such that

$$\prod_{i=0}^{k_j-1} \|Df|_{E^s(f^i(f^{j_j} q))}\| > \lambda^{k_j}$$

for any $j \in \mathbb{N}$. Hence we can take a sequence of positive integers by $K_1 = K_0, K_{i+1} = K_i + k_{K_i}$. Then we take $K_l < K_m$ such that $K_m - K_l$ is divisible by $\pi(q)$. We know that

$$\prod_{i=0}^{K_m-K_l-1} \|Df|_{E^s(f^i(f^{K_l} q))}\| > \lambda^{K_m-K_l}$$

for all $i \in \mathbb{N}$. This contradicts the inequalities in the assumption.

Let $0 \leq n_1 < n_2 < \dots$ be all nature numbers satisfying

$$\prod_{i=0}^{k-1} \|Df|_{E^s(f^i(f^{n_i}q))}\| \leq \lambda^k$$

for all $k \geq 1$. Similarly, let $0 \leq m_1 < m_2 < \dots$ be all nature numbers satisfying

$$\prod_{i=0}^{k-1} \|Df^{-1}|_{E^u(f^{-i}(f^{m_i}q))}\| \leq \lambda^k,$$

for all $k \geq 1$. From the above discussion we know that $\{n_i\}, \{m_i\}$ are all nonempty. We just need to prove that $\{n_i\} \cap \{m_i\} \neq \emptyset$.

If $\{n_i\} \cap \{m_i\} = \emptyset$, then we can find two integers $m_a < n_b$ such that (m_a, n_b) contains no integers which belongs to $\{n_i\}$ and $\{m_i\}$.

Claim. For every $m_a \leq l < n_b$,

$$\prod_{j=l}^{n_b-1} \|Df|_{E^s(f^j(q))}\| > \lambda^{n_b-l}$$

Proof. We will prove the claim by induction. It is easy to check that if $\|Df|_{E^s(f^{n_b-1}(q))}\| \leq \lambda$, then for any $k \geq 1$,

$$\prod_{j=n_b-1}^{n_b-1+k-1} \|Df|_{E^s(f^j(q))}\| \leq \lambda^k.$$

This contradicts $n_b - 1 \notin \{n_i\}$. Hence $\|Df|_{E^s(f^{n_b-1}(q))}\| > \lambda$, therefor the claim is true for $l = n_b - 1$.

Assume the claim is true for $l + 1, \dots, n_b - 1$. Hence

$$\prod_{j=n}^{n_b-1} \|Df|_{E^s(f^j(q))}\| > \lambda^{n_b-n},$$

for any $n \geq l + 1$. If

$$\prod_{j=l}^{n_b-1} \|Df|_{E^s(f^j(q))}\| \leq \lambda^{n_b-k},$$

then

$$\prod_{j=l}^{l+k-1} \|Df|_{E^s(f^j(q))}\| \leq \lambda^k,$$

for any $k > 1$, contradicting with $k \notin \{n_i\}$. Hence

$$\prod_{j=l}^{n_b-1} \|Df|_{E^s(f^j(q))}\| > \lambda^{n_b-l}.$$

This finishes the proof of the claim.

Using the hypothesis that $\|Df|_{E^s(x)}\|/m(Df|_{E^u(x)}) < \lambda^2$ for any $x \in \text{Orb}(q)$, we can get $\|Df^{-1}|_{E^u(fx)}\| < \lambda^2 \|Df|_{E^s(x)}\|^{-1}$. It follows that for any $m_a \leq l < n_b$,

$$\prod_{j=l+1}^{n_b} \|Df^{-1}|_{E^u(f^j(q))}\| < \lambda^{n_b-l}.$$

Hence $n_b \in \{m_i\}$, contradicting $\{n_i\} \cap \{m_i\} = \emptyset$. Hence $\{m_i\} \cap \{n_j\} \neq \emptyset$. By taking $m \in \{m_i\} \cap \{n_j\}$ and $q' = f^m(q)$ we finish the proof of the lemma.

If a hyperbolic periodic point q satisfies

$$\prod_{i=0}^{k-1} \|Df|_{E^s(f^i(q))}\| \leq \lambda^k,$$

$$\prod_{i=0}^{k-1} \|Df^{-1}|_{E^u(f^{-i}(q))}\| \leq \lambda^k$$

for any $k > 0$, we call it a “good” hyperbolic periodic point. It is known that the “good” hyperbolic points always has uniform sized stable and unstable manifolds if it is contained in an invariant set with dominated splitting(See the Corollary 3.3 of [PS2]).

Lemma 4.3. *Assume that $H(p, f)$ satisfies the hypothesis (P1)-(P3) and the bundle E is not contracting. Then for any constants $\lambda < \lambda_1 < \lambda_2 < 1$ and $\delta > 0$, there exist a δ -pseudo orbit $\{x_i\}_{i=0}^{i=n}$ in $H(p, f)$ such that:*

1. $\prod_{i=0}^{k-1} \|Df|_{E(x_i)}\| < \lambda_2^k$ and $\prod_{i=k}^n \|Df^{-1}|_{F(x_i)}\| < \lambda_2^{n-k+1}$, for $k = 1, 2, \dots, n$.
2. $\prod_{i=0}^{n-1} \|Df|_{E(x_i)}\| > \lambda_1^n$.
3. $x_0 = x_n$ is a “good” hyperbolic periodic point.

Proof. Since E is not contracting, one can find a “bad” point b in $H(p, f)$ such that

$$\prod_{j=0}^{n-1} \|Df|_{E(f^j b)}\| \geq 1$$

for all $n \geq 1$.

Since $H(p, f)$ is a homolonic class, there is a hyperbolic periodic point q homoclinically related with p such that $\text{Orb}(q)$ forms a δ -net of Λ , i.e., for every $a \in \Lambda$, there is $x \in \text{Orb}(q)$ such that $d(a, x) < \delta$. By the hypothesis (P2) and lemma 4.2, we can assume that q is a “good” hyperbolic periodic point.

Now we construct the pseudo orbit $\{x_i\}_{i=0}^{i=n}$ by combining the orbit of q with the orbit of b .

Let $x_0 = q, x_1 = f(q), x_2 = f^2(q), \dots, x_{s \cdot \pi(q)} = q$ where s will be chosen big enough. Let $0 < t_1 \leq \pi(q)$ be an integer such that $d(f^{t_1}, b) < \delta$. Then we let $x_{s \cdot \pi(q)+1} = f(q), x_{s \cdot \pi(q)+2} = f^2(q), \dots, x_{s \cdot \pi(q)+t_1-1} = f^{t_1-1}(q), x_{s \cdot \pi(q)+t_1} = b$. By the choice of q we know that

$$\frac{1}{k} \sum_{i=0}^{k-1} \log(\|Df|_{E(x_i)}\|) < \log \lambda < \frac{\log \lambda_1 + \log \lambda_2}{2},$$

for every $1 \leq k \leq s\pi(q) + t_1$.

Choose l such that

$$\frac{1}{s\pi(q) + t_1 + l} \left(\sum_{i=0}^{s\pi(q)+t_1-1} \log(\|Df|_{E(x_i)}\|) + \sum_{i=0}^{l-1} \log(\|Df|_{E(f^i(b))}\|) \right) \geq \frac{\log \lambda_1 + \log \lambda_2}{2},$$

and for any $l' < l$,

$$\frac{1}{s\pi(q) + t_1 + l'} \left(\sum_{i=0}^{s\pi(q)+t_1-1} \log(\|Df|_{E(x_i)}\|) + \sum_{i=0}^{l'-1} \log(\|Df|_{E(f^i(b))}\|) \right) < \frac{\log \lambda_1 + \log \lambda_2}{2}.$$

The existence of l is ensured by the property of b . Then we take $x_{s \cdot \pi(q) + t_1 + 1} = f(b), \dots, x_{s \cdot \pi(q) + t_1 + l - 1} = f^{l-1}(b)$.

Take $0 \leq t_2 < \pi(q)$ such that $d(f^l(b), f^{-t_2}(q)) < \delta$ and $x_{s \cdot \pi(q) + t_1 + l} = f^{-t_2}(q), x_{s \cdot \pi(q) + t_1 + l + 1} = f^{-t_2 + 1}(q), \dots, x_{s \cdot \pi(q) + t_1 + l + t_2} = q$. This ends the construction of $\{x_i\}$. Roughly speaking, the δ -pseudo orbit $\{x_i\}$ is composed by some repeat of $\text{Orb}(q)$ and a segment of $\text{Orb}(b)$.

Denote $S(k) = \sum_{i=0}^{k-1} \log(\|Df|_{E(x_i)}\|)$ and $n = s \cdot \pi(q) + t_1 + l + t_2$. Since $\log(\|Df|_{E(x)}\|)$ have a uniform upper bound and

$$S(s \cdot \pi(q) + t_1 + l) \geq (s \cdot \pi(q) + t_1 + l) \cdot \frac{\log \lambda_1 + \log \lambda_2}{2}.$$

The inequality $S(n) > n \cdot \lambda_1$ will be true by choosing s big enough.

By the construction of $\{x_i\}$,

$$\frac{1}{k} S(k) < \frac{\log \lambda_1 + \log \lambda_2}{2}$$

for any $0 < k < s \cdot \pi(q) + t_1 + l$. Since $\log(\|Df|_{E(x)}\|)$ has a uniform upper bound and $t_2 < \pi(q)$, one can choose s big enough such that for any $0 < k \leq n$, $\frac{1}{k} S(k) < \log \lambda_2$.

Denote $T(k) = \sum_{i=n-k+1}^n \log(\|Df^{-1}|_{F(x_i)}\|)$. Since q is a “good” hyperbolic periodic point,

$$\frac{1}{k} T(k) < \log \lambda (< \log \lambda_2)$$

for any $k = 1, 2, \dots, t_2 + 1$. By the choice of l , we can get that

$$\frac{S(s \cdot \pi(q) + t_1 + l) - S(k)}{s \cdot \pi(q) + t_1 + l - k} > \frac{\log \lambda_1 + \log \lambda_2}{2}$$

for any $k < s \cdot \pi(q) + t_1 + l$. The hypothesis (P1) means that $\log(\|Df^{-1}|_{F(f(x))}\|) < 2 \log \lambda - \log(\|Df|_{E(x)}\|)$. Therefore

$$\frac{T(t_2 + 1 + k) - T(t_2 + 1)}{k} < 2 \log \lambda - \frac{\log \lambda_1 + \log \lambda_2}{2} < \log \lambda$$

for any $0 < k < s \cdot \pi(q) + t_1 + l$. Hence for every $0 < k \leq n$

$$\frac{1}{k} T(k) < \log \lambda < \log \lambda_2.$$

This ends the proof of the lemma.

Now we prove Proposition 4.1. Suppose that $H(p, f)$ satisfies (P1)-(P3) and the bundle E is not contracting. Fix two constants $\lambda < \lambda_1 < \lambda_2 < 1$. From Lemma 4.3 we know that for any $\delta > 0$, there exists a δ -pseudo orbit $\{x_i\}_{i=0}^n$ with the properties in Lemma 4.3. Let $q = x_0 = x_n$ be the “good” periodic point. By the assumption (P3) we know that there exists a periodic point q' close to q such that the orbit of q' shadows the pseudo orbit $\{x_i\}_{i=0}^n$.

If we choose δ small enough, from the properties (1) and (2) in Lemma 4.3 we know that q' is also a “good” periodic point with respect to λ_2 . It follows that q' and q all have uniform sized stable and unstable manifolds, therefor q' will be homoclinic related with q if q' is close to q enough. Hence we can get a $q' \in H(p, f)$ with large period (at least l) such that $\prod_{i=0}^{\pi(q')-1} \|Df|_{E^s(f^i(q'))}\| > \lambda_1^{\pi(q')}$, contradicting hypothesis (P2). This proves Proposition 4.1.

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